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Bireference Procedure fBIP for Interactive Multicriteria Optimization with Fuzzy Coefficients

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Abstract

In the paper an approach to decision making in situations with non-point-like characterisation and subjective evaluation of the actions is considered. The decision situation is represented mathematically as fuzzy multiobjective linear programming (fMOLP) model, where we apply the reduced fuzzy matrices instead of fuzzy classical numbers. The fMOLP model with reduced parameters is decomposable into the set of point-like models and the point-like models enable effective construction of an optimisation procedure – fBIP, see Wojewnik (2006ab), extending the bireference procedure by Michalowski and Szapiro (1992). The approach is applied to a fuzzy optimization problem in the area of telecommunication services.

Keywords: decision support, multicriteria decision making, interactive optimization, fuzzy optimization

JEL Classification: C61, D81, L96.

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1 Problem formulation

We consider a class of decision problems where the point-like description of decision situation is insufficient and therefore we apply the fuzzy set theory. Moreover, we assume that the decision maker modifies her perspective during the problem solving process and therefore we apply the multicriteria interactive approach to identify the final recommendation of the problem solution.
Let us present the motivation to use fuzzy descriptions for economic problems and subjectivity in decision making support.
The economic problems which are characterized by points and vectors in $\mathbb{R}^n$ space admit decision support procedures in precise identification of the problem solution. However, there are decision problems described in natural language, where the single vector representation is not adequate. The inadequacy of the point-like representations results from physiological and psychological features of humans and effects the formulation of economic problems. Human perception provides examples of phenomena when different stimulations result in the same sensor activity. Then these phenomena are labeled with one name (while in point-like description several labels would be used). The sensual sensitivity was examined in the detection experiments from plenty perspectives, e.g.: acoustic and visual perception, see Swets, Green, Getty (1978), Glezer (2009). If the symbol L means the light lumination, a – the color from green to magenta, and b – the color from blue to yellow, then the humanbeing is not able to distinguish two lights different by:

$$E = \sqrt{\Delta L^2 + \Delta a^2 + \Delta b^2},$$

where ΔE is smaller than 1, see e.g. www.cie.co.at.
If one expression is used to describe a few stimulations then Aronson and Wieczorkowska (1999) refer to this fact as to cathegorization. According to earlier findings of Wieczorkowska (1991), the natural language expressions have some width, and thus the points of the $\mathbb{R}^n$-space are inappropriate to represent such non-point-like concepts. Moreover, Wieczorkowska (1991) argues that in some cases the model based on the $\mathbb{R}^n$-points results in false conclusions. For example, let us consider the minimal distance axiom. If the X-stimulant is identified as Y more often than it is identified as X, then the stimulant is more similar to Y than to itself. If the distance will be denoted by $d(\cdot,\cdot)$, then the inequality holds $d(\mathrm{X},\mathrm{Y}) < d(\mathrm{X},\mathrm{X})$, though the measure axiom says $d(\mathrm{X},\mathrm{X}) = 0$ and $d(\cdot,\cdot) \geq 0$.
Let us consider the economic analyses, where the point-like description of concepts is insufficient and the non-point-like terms are used. For example, to describe the macroeconomic situation the economists exploit the expressions like: small inflation, moderate GDP growth, high public expenditures etc., where none of the expressions has a point-like representation (small inflation means both 0,5% and 1%). Though the economic terms sound clear, but the perception of the terms and the following decisions differ among individuals according to their experience, knowledge and

personal situation. Therefore, the individual perspective plays always an important role in economic analysis.

The argument above – the economic examples, as well as the physiological and psychological results – legitimises application of non-point-like values to describe the economic decision problems.

In the paper we take into account also another decision making perspective which is related to the fact that different people solve the same problem in different ways and get different solutions. For example, the stock market investors dispose similar budget and information. Among the criteria they consider are identical ones – profitability and risk – but their portfolios differ and as a consequence they earn and lose to different extend. The situation gets even more complicated if we consider the following two facts. Firstly, the investors regard the future projections of the profit and the risk rather than their current objective values. Secondly, the investors consider a few criteria simultaneously. The profitability is calculated in the form of return, internal rate of return, net present value, and the risk – in the form of price variability, probability of reaching some extreme values, and stock liquidity.

One of the explanations for the individualised result of decision making process is that investors formulate the projections for the future and make the decisions basing on their own experience and social, cognitive and emotional factors, see the seminal paper of Kahneman and Tversky (1979), and the series of articles by Thaler (1987–1990). Therefore, we will support such decision makers, only if we deepen their insight into the decision problem and give them possibility to sovereignly identify the problem solution. In particular we consider the interactive optimization methods, where the phases of information gathering and optimal solution identification are applied alternately. DM evaluates each trial solutions and her opinion is exploited to find the next efficient solution, see e.g. Kaliszewski (2006), Roy (1996), Slowinski (1984), Trzaskalik and Michnik (2002).

Considering the structure of the optimization model: a few partial criteria, feasibility constraints and preference model, Kaliszewski (2004) distinguishes three groups of interactive methods: weighted methods, constraints methods and reference points methods.

The authors of interactive weighted methods assume there is a mathematical structure aggregating the values of partial criteria. In the literature there are various types of aggregating functions: linear, Chebyshev, and augmented Chebyshev of type I and II, see e.g. Wierzbicki (1986), Kaliszewski (2006). The decision maker reviews and evaluates the subsequent efficient solutions. The revealed information changes the weights of the aggregation function.

In the *interactive ε-constraints methods* the multicriteria problem is approximated by a series of single criteria problems. In the auxiliary problems one of the partial criteria is optimized and some additional constraints on the rest of criteria are formulated. Benayoun, Montgolfier, Tereny, Laritchev (1970) in STEM approach propose to identify the trial solutions with Chebyshev metric. The ε-constraints represent the

decision maker's opinion on the expected change in criteria values. Haimes and Hall (1974) in SWT method propose to introduce for every trial solution the substitution function in the Lagrange form. In 1981 Sakawa presented SPOT method where the constraints are modified following the marginal substitution rate between the criteria. Sakawa assumes that the decision maker can formulate the rate for every trial solution. The *reference point methods* employ the ideal and the worst evaluation, to describe the decision maker's aspiration y^* and the reservation level y^-. Conceptually, the final solution should be possibly close to y^*, but in every iteration the metric and the reference points are changed as the decision maker evaluates current solutions, see e.g. Wierzbicki (1982), Jaszkiewicz and Slowinski (1999), Michalowski and Szapiro (1992).
In the paper we apply the fuzzy version of the bireference procedure introduced by Michalowski and Szapiro (1992) and recommended by Luque, Ruiz, Steuer (2010). The procedure is a reference point method, but one can prove its connections to the weighted and ε-constraints methods ex post. For example, given the recommendation in the decision problem the trade-off weights are reckonable.
The paper is organized as follows. The Section 2 presents the concept of fuzzy optimization. We introduce the fuzzy multiobjective linear programming model (fMOLP) to represent the decision situations considered in this paper. We show the properties of presented fMOLP model employed to build the decision support method – the fuzzy bireference procedure in Section 3. Next, in Section 4, we use the procedure to analyze the problem of pricing the telecommunication services. The performance of the introduced method in comparison to other interactive fuzzy optimization methods is discussed in Section 5. The paper ends with Conclusion remarks, Bibliography and Appendix with proofs of theorems presented in Section 2.

2 Decision making and the optimization model

In this paper we consider supporting the decision making process with mathematical models and procedures. In this paragraph we present the point-like model – typically used in optimization – and introduce its fuzzified version – tackling the non-point-like description of the decision problem.

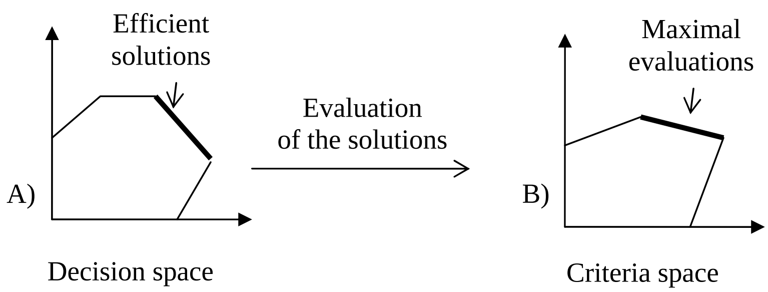
2.1 Point-like model

If the particular decisions, feasibility constraints, decision evaluations, mapping from decisions to evaluations, and preferences are described as vectors in $\mathbb{R}^n$ space, then the decision problem is represented by the point-like model. In particular the decision is represented by the vector $x \in \mathbb{R}^n$, and the result of the decision $x \in \mathbb{R}^n$ – by the vector $y \in \mathbb{R}^m$, with mapping $y = C \cdot x$, and the matrix $C \in \mathbb{R}^{m \times n}$ is called the evaluation matrix. The feasible set is defined by the conditions $A \cdot x \leq b$, where the matrix $A \in \mathbb{R}^{k \times n}$, and the vector $b \in \mathbb{R}^k$ are called respectively technology matrix and the constraint vector. In this case we obtain the *multiobjective linear programming* (MOLP) *problem* of the form

$$\mathrm{C} \cdot \mathrm{x} \to \max, \text{ s.t. } \mathrm{A} \cdot \mathrm{x} \leq \mathrm{b}. \text{ where } \mathrm{x} \in \mathbb{R}^k, \mathrm{A} \in \mathbb{R}^{k \times n}, \mathrm{C} \in \mathbb{R}^{k \times n}, \mathrm{b} \in \mathbb{R}^k.$$

Definition 1. *The* MOLP problem *is defined as the task of finding the set* Y^N *– the C-mappings maximal in the relation "*$\leq$*" in the set* $\mathrm{Y}^D = \mathrm{C} \cdot \mathrm{X}^D$*, where* $\mathrm{X}^D = \{\mathrm{x} : \mathrm{A} \cdot \mathrm{x} \leq \mathrm{b}\}$*. The seventuple* $\sigma = \langle n, m, k, \mathrm{A}, \mathrm{b}, \mathrm{C}, \leq \rangle$*, is called the* structural elements of the MOLP problem*. Every admissible solution* $\mathrm{x}^E \in \mathrm{X}^D$ *with C-mapping* $\mathrm{y}^N = \mathrm{C} \cdot \mathrm{x}^E$ *in the set* Y^N *is called* the efficient solution of the MOLP problem*, see Figure 1.*

Figure 1: Optimization problem



The polygon in A) represents the set of feasible solutions, while the polygon in B) – their evaluations in the criteria space.

If the problem is formulated in the MOLP framework, then the set of admissible solutions is represented by a polygon in $\mathbb{R}^n$ space, see Figure 1A). Similarly in the linear mapping, $\mathrm{y} = \mathrm{C} \cdot \mathrm{x}$, the set of feasible evaluations yields a polygon, too, see Figure 1B). The formulation looks similar to Dantzig linear program but the analysis is much more complicated. In case of a few criteria the maximization problem has no single solution, but a set of differently evaluated Pareto-optimal alternatives, see Figure 1B). Therefore the identification of a distinct solution requires additional information on the preferences of the decision maker that might be established in some interactive approach. Alternately the analyst provides a single Pareto solution and the decision maker evaluates it giving some advices on the expected properties of the next trial solution.

To assure the reliability of the identified solution in this paper we exploit the bireference procedure by Michałowski and Szapiro (1991) verified both theoretically, see Szapiro (1993) and empirically; see Michnik (2000), Polak and Szapiro (2001) and Wojewnik (2006a), (2006b).

2.2 Bi-reference procedure

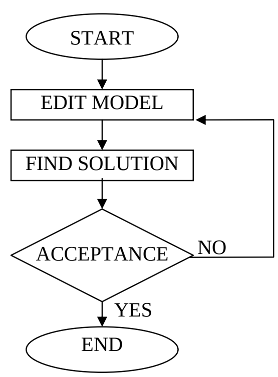
If the analyst will be supplied with the $(\mathrm{A}, \mathrm{b}, \mathrm{C})$-model of the decision situation, then the bi-reference procedure is a tool to identify the admissible trial solutions successively. The solutions follow the preferences of the decision maker and they will be identified until the decision maker is satisfied or it is impossible to improve. The stages of the bireference procedure might be described as follows, see Figure 1.
The procedure begins with *MODEL EDITION*, where the decision maker provides the seventuple $\sigma = \langle n, m, k, \mathrm{A}, \mathrm{b}, \mathrm{C}, \leq \rangle$, $n, m, k \in \mathbb{Z}$, $\mathrm{A} \in \mathbb{R}^{k \times n}$, $\mathrm{b} \in \mathbb{R}^{k}$, $\mathrm{C} \in \mathbb{R}^{m \times n}$, see Def.1. Let $\mathrm{x} \in \mathbb{R}^{n}$ stand for the decision and $\mathrm{y} \in \mathrm{Y} \subset \mathbb{R}^{m}$ for its valuation. The decision maker supplies also the distance in the criteria space $\mathrm{e} \in \mathbb{R}^{m}$ indistinguishable to her. If the worst accepted outcome $\mathrm{y}_W \in \mathbb{R}^{m}$ and the most preferred evaluation $\mathrm{y}_U \in \mathbb{R}^{m}$ are not supplied, they can be calculated as:

$$\forall_{p=1\ldots m} \;\; \mathrm{y}_{W_p} = \min \left\{ \sum_i \mathrm{c}_{pi}\mathrm{x}_i | \mathrm{x} \in \mathrm{X} \right\},$$

$$\forall_{p=1\ldots m} \;\; \mathrm{y}_{U_p} = \max \left\{ \sum_i \mathrm{c}_{pi}\mathrm{x}_i | \mathrm{x} \in \mathrm{X} \right\}.$$

The reference points play crucial role in the *FIND SOLUTION* step and their

Figure 2: Scheme of the interactive decision making procedure



values do change during the optimization proces. In r-th iteration the points $\mathrm{y}_W(r)$ and $\mathrm{y}_U(r)$ depend on the previous reference points and the decisions over the last trial solution $\mathrm{y}_T(r-1)$. For every component of the trial solution the decision maker suggests bettering, leaving at current value or worsening, and thus the indices of the criteria are divided respectively to the sets $I^+(r)$, $I^0(r)$ and $I^-(r)$, where $I^+(0) = \{1, 2, \ldots, m\}$, $I^0(0) = \varnothing$ and $I^-(r) = \varnothing$ at the beginning. Then the extreme values are displaced as follows:

$$\forall_{p=I^{+}(r)}\ \ \mathrm{y}_{W_p}(r) = \mathrm{y}_{T_p}(r-1)$$

$$\forall_{p=I^{0}(r)}\ \ \mathrm{y}_{W_p}(r) = \mathrm{y}_{U_p}(r) = \mathrm{y}_{T_p}(r-1)$$

$$\forall_{p=I^{-}(r)}\ \ \mathrm{y}_{W_p}(r) = \mathrm{y}_{W_p}(0)$$

The set of currently admissible outcomes is defined as:

$$Y(r) = \left\{\mathrm{y} \in \mathrm{Y} | \forall_{p=I^{0}(r)}\ \ f_p(\mathrm{x}) = \mathrm{y}_{T_p}(r-1)\right\}$$

Next, $y_{AW}(r)$ – the admissible outcome closest to $y_W(r)$ – is found:

$$\mathrm{y}_{AW}(r) = \underset{\mathrm{y}\in\mathrm{Y}(r)}{\arg\min}\|\mathrm{y}_W(r) - \mathrm{y}\|.$$

Then the improvement direction:

$$d(r) = \mathrm{y}_U(r) - \mathrm{y}_{AW}(r)$$

and admissible trial solution is calculated:

$$\mathrm{y}_{AT}(r) = \max\left\{\mathrm{y}_{AW}(r) + t\cdot d(r) | t > 0,\ \mathrm{y}_{AW}(r) + t\cdot d(r) \in \mathrm{Y}(r)\right\}.$$

If $\mathrm{y}_{AT}(r)$ is dominated, then the improvement direction d(r) is projected on the closest admissible hyperplane to find the non-dominated outcome $\mathrm{y}_T(r)$.
If the trial solution is ACCEPTED or it does not differ from the previous one $|\mathrm{y}_T(r-1) - \mathrm{y}_T(r-1)| < \mathrm{e}$, then the procedure ENDS. In other case the decision maker betters, leaves at current value or worsens the particular components of current trial solution and thus MODEL EDITION starts over again.

2.3 Fuzzy model

Before we formulate the fuzzy multiobjective linear programming problem (fMOLP) let us present the notation of fuzzy sets and fuzzy relations used in the paper:

1. $x = \{(\mathrm{x}, \mu_x(\mathrm{x})), \mathrm{x} \in \mathbb{R}^n, \mu_x : \mathbb{R}^n \to [0,1]\},\ x \in \mathcal{F}(\mathbb{R}^n)$ – fuzzy set (italics denote fuzzy sets),
2. $x \in \mathcal{F}(\mathbb{R}^n),\ A \in \mathcal{F}(\mathbb{R}^{k\times n}),\ C \in \mathcal{F}(\mathbb{R}^{k\times n}),\ b \in \mathcal{F}(\mathbb{R}^k)$ – fuzzy vectors and matrices,
3. "$\leq$" $\in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n)$ – fuzzy inequality relation:

$$\mu_{\leq}(a,b) = \begin{cases} \min\{\mu_a(\mathrm{a}), \mu_b(\mathrm{b})\} & \text{for } \mathrm{b} - \mathrm{a} \in \mathbb{R}^n_+ \\ 0 & \text{for } \mathrm{b} - \mathrm{a} \in \mathbb{R}^n \backslash \mathbb{R}^n_+ \end{cases}$$

4. $A+B=C \Leftrightarrow \mu_C(\mathrm{C}) = \sup\limits_{\mathrm{A,B}\in\mathbb{R}^{m\times n}} \{\min\{\mu_A(\mathrm{A}), \mu_B(\mathrm{B})\} \,|\mathrm{A}+\mathrm{B}=\mathrm{C}\}$ – sum of fuzzy numbers,
5. $A\cdot D = E \Leftrightarrow \mu_E(\mathrm{E}) = \sup\limits_{\mathrm{A}\in\mathbb{R}^{m\times n},\mathrm{D}\in\mathbb{R}^{n\times k}} \{\min\{\mu_A(\mathrm{A}), \mu_D(\mathrm{D})\} \,|\mathrm{A}\cdot\mathrm{D}=\mathrm{E}\}$ – multiplication of fuzzy numbers,
6. $X^D \in \mathcal{F}(\mathbb{R}^n),\ \mu_{X^D}(\mathrm{x}) = \sup\limits_{\mathrm{A}\in A[\alpha],\mathrm{b}\in b[\alpha],\mathrm{x}\in\mathbb{R}^n} \{\alpha:\ \mathrm{A}\cdot\mathrm{x}\leq\mathrm{b}\},\ \mathrm{supp}(X^D)\neq\varnothing$ – fuzzy feasible set.

The fuzzy sets are able to represent the non-point-like expressions. For example, the fuzzy set $\mathrm{b}\in\mathcal{F}(\mathbb{R}),\ \mu_b(\mathrm{b}) = \max\left(0, 1-|5-\frac{1}{2}\cdot b|\right)$, represents the value – about 10 – the maximal post-paid minute charge in the telecommunication problem. The interpretation is as follows – the decision maker regards the value of 0,10 PLN as the limit with the full confidence, but she also considers the value of 0,11 PLN with the confidence smaller by a half.

Definition 2. *The task of finding the set Y^N – the C-mappings maximal in the relation "$\leq$" in the set $Y^D = C\cdot X^D$ – is called a* fuzzy multiobjective linear programming (fMOLP) problem. *The seventuple $\sigma^F_{AbC} = \langle n, m, k, A, b, C, \leq\rangle$ is called* the set of structural elements in the fMOLP problem. *Every admissible solution $x^E\in X^D$ with C-mapping $y^N = C\cdot x^E$ in the set Y^N is called* the efficient solution of the fMOLP problem.

The fMOLP problem is introduced to formulate the non-point-like description for the decision problems. As an example let us consider the typical problem of tariff setting in the telecom industry.

<u>Problem 1</u> The board of directors wants to maximize the earnings, both in the prepaid and postpaid market.

The postpaid clients sign the time-limited contract and they pay the fixed amount of money for the network access and some variable amount – for the voice connections, where the minute charge is rather low. The postpaid clients are tied by the contract and thus they assure some fixed income for the company.
The pre-paid customers do not sign the contract and they can leave the company at any moment. Moreover, they do not pay any fixed but the variable amount only. Generally, comparing to the postpaid customer the prepaid customer are less profitable and more risky. However, the prepaid clients are usually younger and more dynamic and they are expected to grow in the future and to be more apt to up- and cross-selling.
Although the Problem 1 sounds clear in terms of economic phrases but the introduction of the mathematical MOLP model is troublesome, because the point-like characterization of the decision situation is required. In particular, the precise numeric specification of competition decisions, unit costs or the demand reaction to

the price changes does not include the element of expectation and uncertainty that is present in expert opinions. First of all, it is impossible to determine whether 0,1 PLN or 0,09999 PLN should be used for the average minute cost. Second, describing the influence of the price on the demand we can not be sure as to the functional form of the dependency. As a result the telecom problem requires the non-point-like description and the fMOLP problem introduces such description. Whilst the formulation of the fuzzy parameters, especially the fuzzy matrix, seems to be a complex task, we introduce the reduced fuzzy matrix.

Definition 3. *The fuzzy matrix $A \in \mathcal{F}(\mathbb{R}^{m\times n})$ is called a* fuzzy matrix of reduced support*, or simply* – reduced fuzzy matrix – *if the membership function is given by:*

$$\mu_A(\mathrm{A}) = \begin{cases} f(t) & \text{for}\, \mathrm{A} = (1-t)\cdot \mathrm{A}^L + t\cdot \mathrm{A}^U,\, t\in[0,1] \\ 0 & \text{in other case} \end{cases},$$

where $\mathrm{A}^L, \mathrm{A}^U \in \mathbb{R}^{m\times n}$, the function $f:[0,1]\to[0,1]$ is quasi-concave, $f(0)=0$, $f(1)=0$, and $\sup_{t\in\mathbb{R}} f(t)=1$. The reduced fuzzy matrix A is given by the triplet $\left(\mathrm{A}^L, \mathrm{A}^U, f\right)$.

The reduced fuzzy matrices are only a class of fuzzy matrices, but there are situations where the reduced form is sufficient. The situations are defined by linear dependency of the parameter values resulting from technology limitations (e.g. carbon to iron proportion in machine steel) or law regulations (e.g. earnings and taxes).
In the following we present the properties of reduced fuzzy matrices exploited in fBIP procedure.
Every α-cut of the reduced fuzzy matrix $A \in \mathcal{F}(\mathbb{R}^{m\times n})$ given by triplet $\left(\mathrm{A}^L, \mathrm{A}^U, f\right)$ is a section $({}_-A[\alpha], {}^-A[\alpha])$, where

$${}_-A[\alpha] = \mathrm{A}^L + \left(\mathrm{A}^U - \mathrm{A}^L\right)\cdot \min f^{-1}(\alpha)$$

and

$${}^-A[\alpha] = \mathrm{A}^L + \left(\mathrm{A}^U - \mathrm{A}^L\right)\cdot \max f^{-1}(\alpha).$$

The reduced fuzzy matrix $A^r \in \mathcal{F}(\mathbb{R}^{m\times n})$ defined by the triple $\left(\mathrm{A}^L, \mathrm{A}^U, f\right)$ fuzzifies the point-like matrix $\mathrm{A} \in \mathbb{R}^{m\times n}$ only in the direction $\mathrm{d} = \mathrm{A}^L - \mathrm{A}^U$, while the fuzzy matrix $A \in \mathcal{F}(\mathbb{R}^{m\times n})$ – in any direction $\mathrm{d} \in \mathbb{R}^{m\times n}$. However, the reduced fuzzy matrices in some situations perform in the same way as the fully fuzzified matrices, see Theorem 1.

Theorem 1 (on the decomposition of fuzzy conditions with reduced parameters)**.** *The singleton $S(\mathrm{x},\alpha) \in X \subset \mathcal{F}(\mathbb{R}^n_+)$ solves the inequality*

$$A\cdot x \le b,$$

if and only if it solves the set of inequalities:

$$\begin{cases} a_1^T \cdot x \le b_1 \\ \quad \vdots \\ a_k^T \cdot x \le b_k \end{cases},$$

where the reduced fuzzy vectors $a_j \in \mathcal{F}(\mathbb{R}^n)$ *are given by the triplets* $\left(\mathrm{a}_j^L, \mathrm{a}_j^U \in \mathbb{R}^n, f_A : [0,1] \to [0,1]\right)$, $\mathrm{a}_j^L \le \mathrm{a}_j^U$, *the fuzzy numbers* $b_j \in \mathcal{F}(\mathbb{R})$ *– by the triplets* $\left(\mathrm{b}_i^L, \mathrm{b}_i^U \in \mathbb{R}, f_b : [0,1] \to [0,1]\right)$, $j = 1, \ldots, k$, *and the reduced fuzzy matrix* $A \in \mathcal{F}(\mathbb{R}^{k\times n})$ *and the reduced fuzzy vector* $b \in \mathcal{F}(\mathbb{R}^k)$ *– respectively by the triplets* $\left(\mathrm{A}^L = [\mathrm{a}_{ij}^L], \mathrm{A}^U = [\mathrm{a}_{ij}^U], f_A : [0,1] \to [0,1]\right)$ *and* $\left(\mathrm{b}^L = [\mathrm{b}_i^L], \mathrm{b}^U = [\mathrm{b}_i^U], f_b : [0,1] \to [0,1]\right)$.

From the Theorem 1 we can conclude that the set of inequalities, $a_j^T \cdot x \le b_j$, $j = 1, \ldots, k$, parametrized by the fuzzy numbers of the same membership functions ($f_A : [0,1] \to [0,1]$ for a_j, $j = 1, \ldots, k$, and $f_b : [0,1] \to [0,1]$ for b_j, $j = 1, \ldots, k$, e.g. triangular, Gaussian, trapezoidal, linear splines) has the solution set equal to the solution set of inequality, $A \cdot x \le b$, given by the reduced fuzzy matrices.
Let us consider two fMOLP problems given by fuzzy and reduced fuzzy matrices. To define the first fMOLP the user has to introduce $nm + nk + k$ membership functions (or three multiattribute functions $f_A : \mathbb{R}^{n\times k} \to [0,1]$, $f_b : \mathbb{R}^k \to [0,1]$, $f_C : \mathbb{R}^{n\times m} \to [0,1]$). To define the second fMOLP the user has to introduce three membership functions only: $f_A : [0,1] \to [0,1]$, $f_b : [0,1] \to [0,1]$, $f_C : [0,1] \to [0,1]$. Thus the reduced form of the model is much easier tractable than the fully fuzzified optimization problem. Moreover, the solution of the reduced model is decomposable into the set of solutions from the point-like models, see Theorem 2.

Theorem 2. On the decomposition of the fMOLP problem. *Assumptions*
Let us consider the fMOLP problem given by the seventuple $\sigma_{AbC}^F = \langle n, m, k, A, b, C, \le\rangle$, where $n, m, k \ge 1$ are integers, the reduced fuzzy matrix $A \in \mathcal{F}(\mathbb{R}^{k\times n})$ is given by the triplet $\left(\mathrm{A}^L, \mathrm{A}^U \in \mathbb{R}^{k\times n}, f : [0,1] \to [0,1]\right)$, $\mathrm{A}^L \le \mathrm{A}^U$, the reduced fuzzy vector $b \in \mathcal{F}(\mathbb{R}^k)$ - by the triplet $\left(\mathrm{b}^L, \mathrm{b}^U \in \mathbb{R}^{k\times n}, g : [0,1] \to [0,1]\right)$, $\mathrm{b}^L \le \mathrm{b}^U$, and the reduced fuzzy matrix $C \in \mathcal{F}(\mathbb{R}^{m\times n})$ - by the triplet $\left(\mathrm{C}^L, \mathrm{C}^U \in \mathbb{R}^{m\times n}, h : [0,1] \to [0,1]\right)$.
Thesis
(T1) For the fMOLP$_{AbC}$ problem given by the seventuple $\sigma_{AbC}^F = \langle n, m, k, A, b, C, \le\rangle$ there exist a series of the sharpened fMOLP$_{Ab\mathrm{C}}$, $\mathrm{C} \in \mathrm{supp}(C)$, problems given by the seventuples $\sigma_{AbC}^F = \langle n, m, k, A, b, \mathrm{C}, \le\rangle$, that the set of efficient solutions X_{AbC}^E is a sum of the sets $\left\{X_{Ab\mathrm{C}}^E\right\}_{\mathrm{C}\in\mathrm{supp}(C)}$:

$$\mu_{X_{AbC}^E}(\mathrm{x}) = \max_{\mathrm{C}\in C_{[\varepsilon]}} \left(\min\left(\mu_{X_{Ab\mathrm{C}}^E}(\mathrm{x}), \varepsilon\right)\right).$$

(T2) For the fMOLP$_{AbC}$ problem given by the seventuple $\sigma_{AbC}^F = \langle n, m, k, A, b, C, \le\rangle$ there exist a series of the sharpened fMOLP$_{_A_{[\varepsilon]}bC}$, $\varepsilon \in (0,1]$, problems given by the seventuples $\sigma^F_{AbC} = \langle n, m, k, {}_{-}A_{[\varepsilon]}, b, C, \leq \rangle$, that the set of efficient solutions X^E_{AbC} is a sum of the sets $\left\{ X^E_{-A_{[\varepsilon]}bC} \right\}_{\varepsilon \in (0,1]}$:

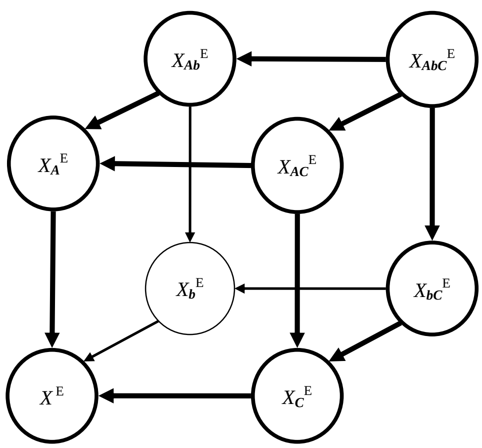
$$\mu_{X^E_{AbC}}(\mathrm{x}) = \max_{\varepsilon \in (0,1]} \left(\min \left(\mu_{X^E_{-A_{[\varepsilon]}bC}}(\mathrm{x}), \varepsilon \right) \right).$$

(T3) For the fMOLP$_{AbC}$ problem given by the seventuple $\sigma^F_{AbC} = \langle n, m, k, A, b, C, \leq \rangle$ there exist a series of the sharpened fMOLP$_{A^-b_{[\varepsilon]}C}$, $\varepsilon \in (0, 1]$, problems given by the seventuples $\sigma^F_{AbC} = \langle n, m, k, A, {}^{-}b_{[\varepsilon]}, C, \leq \rangle$, that the set of efficient solutions X^E_{AbC} is a sum of the sets $\left\{ X^E_{A^-b_{[\varepsilon]}C} \right\}_{\varepsilon \in (0,1]}$:

$$\mu_{X^E_{AbC}}(\mathrm{x}) = \max_{\varepsilon \in (0,1]} \left(\min \left(\mu_{X^E_{A^-b_{[\varepsilon]}C}}(\mathrm{x}), \varepsilon \right) \right).$$

From Theorem 2 we can conclude, that the solution of the fMOLP problem can be found by summing up the solutions of sharpened fMOLP problems, see Figure 3. The Theorems 1-2 enable decomposition of the fMOLP problem into a series of

Figure 3: Decomposition of the efficient solutions set for reduced fuzzy parameters



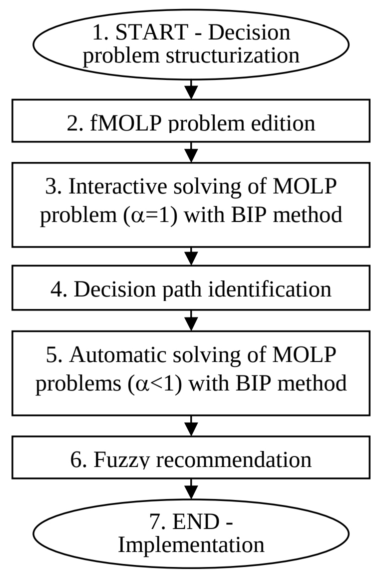
Notes: The symbol X^E_{AbC} denotes the set of efficient solutions in the fMOLP problem with fuzzy AbC. The arrow from X^E_{AbC} to X^E_{AbC} means that it is possible to find series of fMOLP problems with fuzzy AC that the efficient solutions X^E_{AbC} are the sum of X^E_{AbC} solutions, see Theorem 2.

computationally tractable MOLP problems. Therefore it is possible to exploit the properties of the point-like models in the fuzzy formulation.

3 Fuzzy bireference procedure

The process of problem structurization leads to better understanding of the decision situation. If the problem is restructurized a few times, then the insight of the decision maker is even deeper. The fact is exploited in the *interactive decision support methods*. One recognizes two phases of mathematical support: (1) formulation of the optimization model by the decision maker, (2) identification of the optimization model solution by the analyst. In the interactive decision support the phases are performed alternately, the decision maker has the opportunity to observe the results of the problem structure changes and to modify the model accordingly. In this paper we extend the interactive bireference procedure by Michalowski and Szapiro (1992) to find the efficient solutions in the fMOLP problem.
The fuzzy bireference procedure follows the seven steps presented in Figure 4. In the

Figure 4: Fuzzy bireference procedure



Notes: For $\alpha = 1$ in step 3 the bireference procedure BIP by Michalowski and Szapiro (1992) is used interactively with the decision maker, while for $\alpha < 1$ in step 5 - it is used automatically.

phase *START – Decision problem structurization* the decision maker identifies and describes the decision problem by linguistic formulation of structural elements like: the goals, decisions, constraints, the evaluation methods etc.
Following, in the step *fMOLP problem edition* the decision maker supported by the analyst constructs structural elements of the fuzzy multiobjective linear programming problem fMOLP$_{ABC}$, $\sigma^F_{AbC} = \langle n, m, k, A, b, C, \leq \rangle$, as the representation of the

decision problem, and especially the admissible decisions, decision evaluations and the evaluations ordering.

Next, in the step *Interactive solving the MOLP problem ($\alpha = 1$) with BIP method* the analyst identifies the point-like matrices and vectors of the membership $\alpha = 1$:

$$_{-}A_{[1]} = \mathrm{A}^L + \left(\mathrm{A}^U - \mathrm{A}^L\right) \cdot \min f^{-1}(1),$$

$$^{-}b_{[1]} = \mathrm{b}^L + \left(\mathrm{b}^U - \mathrm{b}^L\right) \cdot \max g^{-1}(1).$$

Then the $\mathrm{MOLP}_{1,\mathrm{C}}$ problem of the elements $\sigma_{1,\mathrm{C}} = \langle n, m, k, _{-}A_{[1]}, ^{-}b_{[1]}, \mathrm{C}, \leq\rangle$, where $\mathrm{C} \in C_{[1]}$, is constructed. The decision maker identifies in the $\mathrm{MOLP}_{1,\mathrm{C}}$ problem the most preferred solution using the interactive bireference procedure BIP, see Michalowski and Szapiro (1992).

In the fourth phase, *Decision path identification*, the analyst (or computer system) acquires the indescernibility value ε and the series of decisions $\left(I^+, I^-, I^0\right)_{[1]}$ formulated by the decision maker on the evaluations (I^+ – to make better, I^- – to worsen, I^0 – to leave unchanged the particular criterion) during the $\mathrm{MOLP}_{1,\mathrm{C}}$ problem solving with BIP method.

Next, in the step *Automatic solving of MOLP problems ($\alpha \neq 1$) with BIP method*, the series of MOLP problems with the structural elements $\sigma_{\alpha,\mathrm{C}} = \langle n, m, k, A_{[\alpha]}, ^{-}b_{[\alpha]}, \mathrm{C}, \leq\rangle$, $\mathrm{C} \in C_{[\alpha]}$, $\alpha \in (0,1)$, is identified and the analyst employs the decision path $\left(I^+, I^-, I^0\right)_{[1]}$ to find efficient solutions evaluated at the level $[\mathrm{y}_T]_{\alpha,\mathrm{C}} \in \mathbb{R}^m$.

In the sixth phase the *Fuzzy recommendation* is formulated, where the analyst merges the solutions of the MOLP problems into fuzzy set with C-mapping y^E, where

$$\mu_{y^E}(\mathrm{y}) = \sup\left\{\alpha | \exists_{\mathrm{C}\in C_{[\alpha]}}\, \mathrm{y} = [\mathrm{y}_T]_{\alpha,\mathrm{C}}\right\}.$$

The final step *END - Implementation* includes deployment of the recommendation obtained by the decision maker.

If the reduced fuzzy matrices $A \in \mathcal{F}(\mathbb{R}^{k\times n})$, $b \in \mathcal{F}(\mathbb{R}^k)$, $C \in \mathcal{F}(\mathbb{R}^{m\times n})$ have continuous membership functions $f, g, h : \mathbb{R} \to [0,1]$, then, in general, the number of MOLP problems is infinite. We propose to use the linear splines as membership functions, because the limited number of points is sufficient to represent the functions. The membership function of the solution in the fMOLP problem is interpolated between the results of MOLP problems.

If the fMOLP problem represents the decision problem, then the fuzzy bireference procedure helps the decision maker to interactively identify the recommended solution. The procedure does not require high mathematical competence but still it operates according to the preferences of the decision maker. In the following paragraph we will present the practical telecommunication problem and solve it with fuzzy bireference procedure.

4 Pricing the telecommunication services with fuzzy decision support

To give an overview of the procedure functioning let us consider the Problem 1 - pricing the telecommunication services. In particular market situation (including the competitive companies, service cost, price elasticity of the demand etc.) the total criteria for the best choice is the maximizing of joint post paid and pre-paid earnings. Let us assume 3 million postpaid customers are contracted for 100 PLN fixed payment and 0,8 PLN per minute over the agreed 100 minutes (usually the clients need circa 120 minutes). As a result of contract termination and the non-payment churn about 6,6% of these people leave the company every year, while 200 000 will come as effect of marketing campaigns. The new clients will obtain the conditions at least as good as the old ones.
In the prepaid market the company does not posses any clients at the moment, but it starts the new product and expects about 100 000 people. Each client will present a demand for voice connection of 20 minutes at 0,8 PLN per minute. Whereas, the minute cost exceeds the technical and administrative cost of minute – 0,2 PLN.
To perform as a cheap brand the operator wants to keep the expenses of typical family (one postpaid and two prepaids) under 120 PLN. The general cost of operations in the following quarter of the year is more or less constant at: 20 millions PLN for administration, 100 millions PLN for physical network maintenance and development and 80 millions PLN for the advertising, product development and client service.
Presented description does not reveal a few limitations existent in real telecommunication companies: financial operations, various payment plans for private and business, seasonal sales, price elasticity of talk duration and the other products of the company (e.g. short message system, data transfer). However, these features do not reveal any additional properties of the fuzzy bireference procedure and therefore they are not considered in this paper.
Let us assume the follwoing interpretation:
x_1 – fixed payment in postpaid contracts, x_2 – one minute charge in postpaid considerations, x_3 – one minute charge in prepaid considerations.
If the average number of new postpaid clients is 100k ($\frac{1}{2}$ from 200k), the average number of new prepaid clients is 50k ($\frac{1}{2}$ from 100k), and the average number of old postpaid clients is 2,9M (100%-$\frac{1}{2}$6,6% from 3M) with the avarage earnings of 50 PLN per client, then the maximization of Total Earnings of the company TE might be represented by the criterion:

$$TE = 100k \cdot \mathrm{x}_1 + 100k \cdot 20 \cdot \mathrm{x}_2 + 50k \cdot 20 \cdot \mathrm{x}_3 + 2.9M \cdot 50PLN \to \max$$

or

$$TE' = 0,1 \cdot \mathrm{x}_1 + 2 \cdot \mathrm{x}_2 + 1 \cdot \mathrm{x}_3 \to \max .$$

In the same manner we might introduce the criterion describing the fixed earnings from the new postpaid clients:

$$0,1 \cdot \mathrm{x}_1 \rightarrow \max,$$

and the criterion describing the earnings from the new prepaid clients:

$$1 \cdot \mathrm{x}_3 \rightarrow \max .$$

Assuming ambiguity as to the price elasticity, we might expect the parameters standing in the criteria functions at x_2 and x_3 to be the subject of non-point description. To describe them we will use the fuzzy sets in the following.
The presented problem enables also formulation of the admissible range for every decision variable. The fixed postpaid amount should be in the range [0; 100 PLN] and the postpaid and prepaid minute charge – in [0; 0,8PLN], whereas the upper bounds are subject to non-point descritpion and therefore will be characterized with fuzzy matrices. The least condition covers aspect of cheap, family pricing, e.g.:

$$1 \cdot \mathrm{x}_1 + 20 \cdot \mathrm{x}_2 + 40 \cdot \mathrm{x}_3 \leq 130,$$

where the upper bounds is also subject to fuzzy descritpion.
The following fMOLP model might represent the tariff setting problem, if the structural elements $\sigma^F_{AbC} = \langle 3, 7, 3, A, b, C, \leq \rangle$ are defined with the reduced fuzzy matrices $A \in \mathcal{F}(\mathbb{R}^{7\times 3})$, $b \in \mathcal{F}(\mathbb{R}^{7})$, $C \in \mathcal{F}(\mathbb{R}^{3\times 3})$:

$$C^L = \begin{bmatrix} 0,1 & 1,8 & 0,9 \\ 0,1 & 0 & 0 \\ 0 & 0 & 0,9 \end{bmatrix}, C^U = \begin{bmatrix} 0,1 & 2,2 & 1,1 \\ 0,1 & 0 & 0 \\ 0 & 0 & 1,1 \end{bmatrix},$$

$$\left(A^L\right)^T = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & -1 & 0 & 0 & 1 & 36 \end{bmatrix},$$

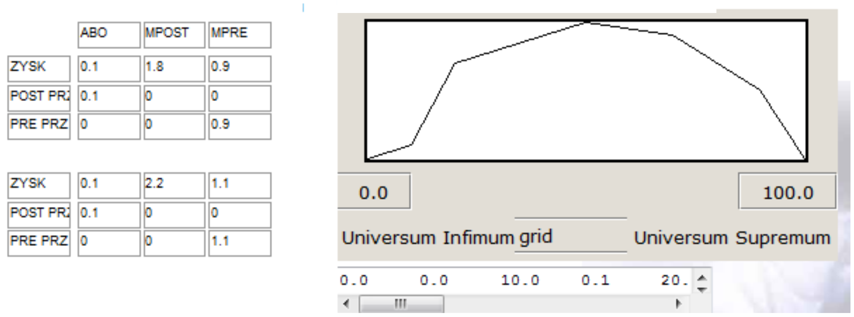
$$\left(A^U\right)^T = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 22 \\ 0 & 0 & -1 & 0 & 0 & 1 & 44 \end{bmatrix},$$

$$\left(b^L\right)^T = \begin{bmatrix} 0 & 0 & 0 & 90 & 0,75 & 0,75 & 120 \end{bmatrix},$$

$$\left(b^U\right)^T = \begin{bmatrix} 0 & 0 & 0 & 110 & 0,85 & 0,85 & 140 \end{bmatrix},$$

and the membership functions are linear splines. The fuzzyfied elements are $c_{1,2}$, $c_{1,3}$, $c_{2,3}$, $c_{3,3}$, $a_{7,1}$, $a_{7,2}$, $a_{7,3}$, b_4, b_5, b_6, b_7.
The process of solving the fMOLP problem starts with formulation of the fuzzy parameter values – for the fuzzy criteria matrix see Figure 5. Next, the

Figure 5: Parameter formulation - printscreen



Notes: To define the fuzzy C matrix the decision maker inputs the left and the right edge of the fuzzy set support C^L, C^U, and the membership function $\mu_C : \mathbb{R} \to [0,1]$.

system identifies the point-like matrices of the highest membership value, $\alpha = 1$, and uses them to construct the point-like $\mathrm{MOLP}_{1,\mathrm{C}}$ problem of the elements $\sigma_{1,\mathrm{C}} = \langle n, m, k, _-A[1], ^-b[1], \mathrm{C}, \leq \rangle$, where $\mathrm{C} \in C[1]$, where

$$C_{[1]} = \begin{bmatrix} 0,1 & 2 & 1 \\ 0,1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left(_-A_{[1]}\right)^T = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & -1 & 0 & 0 & 1 & 40 \end{bmatrix},$$

$$\left(^-b_{[1]}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 100 & 0,8 & 0,8 & 120 \end{bmatrix}.$$

For the problem the reference points are identified, see Figure 6:

$$\mathrm{y}_U(1) = \begin{bmatrix} 11,4 & 10 & 0,8 \end{bmatrix}^T, \ \mathrm{y}_W(1) = \begin{bmatrix} 0,6 & 0 & 0,2 \end{bmatrix}^T,$$

what means, that for the first criterion the extreme admissible values (no matter on other criteria) are 11,4 and 0,6. While $\mathrm{y}_W(1)$ is feasible, then the admissible worst evaluation is $\mathrm{y}_{AW}(1) = \mathrm{y}_W(1)$, and the improvement direction is given by, see Figure 6:

$$\mathrm{d}(1) = \mathrm{y}_U(1) - \mathrm{y}_{AW}(1) = \begin{bmatrix} 10,8 & 10 & 0,6 \end{bmatrix}^T.$$

Maximization of:

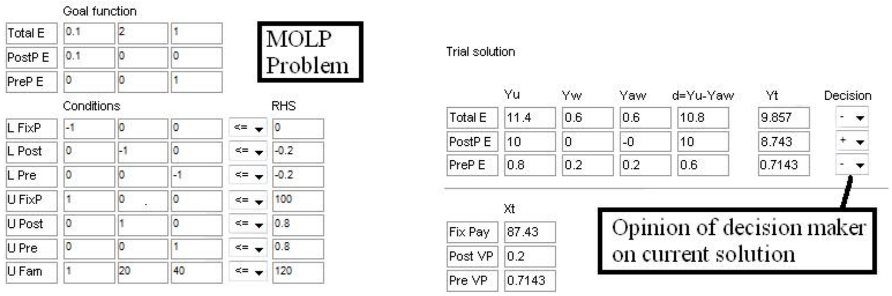
$$\mathrm{y}_{AT}(1) = \max\{\mathrm{y}_{AW}(1) + t \cdot \mathrm{d}(1) | t > 0, \, \mathrm{y}_{AW}(1) + t \cdot \mathrm{d}(1) \in \mathrm{Y}(1)\},$$

where

$$\mathrm{Y}(1) = \{\mathrm{y} \in \mathbb{R}^m | \mathrm{y} = \mathrm{C}_{[1]} \cdot \mathrm{x}, \ _-A_{[1]} \cdot \mathrm{x} \leq ^- b_{[1]}\}$$

gives the first trial solution $x_T(1) = \begin{bmatrix} 87,43 & 0,2 & 0,71 \end{bmatrix}^T$ evaluated at $y_{AT}(1) = \begin{bmatrix} 9,86 & 8,74 & 0,71 \end{bmatrix}^T$, see Figure 6. The trial solution is presented to the decision maker and she chooses the second criterion to be bettered, and the first and third criterion – to be worsened, thus $I^+(1) = \{2\}$, $I^-(1) = \{1,3\}$, $I^0(1) = \varnothing$, see Figure 6. The information supplied by the decision maker enables system to change

Figure 6: DM evaluates the trial solution – printscreen



Notes: For the trial solution of MOLP problem – vector x_T – the evaluation is calculated – vector $y_T = C \cdot x_T$ – and decision maker chooses the elements to be bettered (+), worsened (-) or left unchanged (0).

the problem structure (the reference points) to:

$$y_U(2) = \begin{bmatrix} 11,4 & 10 & 0,8 \end{bmatrix}^T, \; y_W(2) = \begin{bmatrix} 0,6 & 10 & 0,2 \end{bmatrix}^T,$$

and to identify the subsequent trial solution in the same way. The decision maker accepts the proposal or once again chooses the criteria to change.

Let us assume, that the decision maker accepts the third trial solution $x_T(3) = \begin{bmatrix} 95,2 & 0,2 & 0,52 \end{bmatrix}^T$, where $y_{AT}(3) = \begin{bmatrix} 10,44 & 9,52 & 0,52 \end{bmatrix}^T$ following the path $I^+ = (\{2\},\{1,2\})$, $I^- = (\{1,3\},\{3\})$ and $I^0 = (\{\varnothing\},\{\varnothing\})$. Then the system solves the other point-like subproblems $\mathrm{MOLP}_{\alpha<1}$. For example, basing on the fuzzy matirces we can identify the $\mathrm{MOLP}_{\alpha=0,9}$ problem of the elements $\sigma_{0,9,\mathrm{C}} = \langle n, m, k, {}_{-}A[1], {}^{-}b[1], \mathrm{C}_{[0,9]}, \leq \rangle$, where:

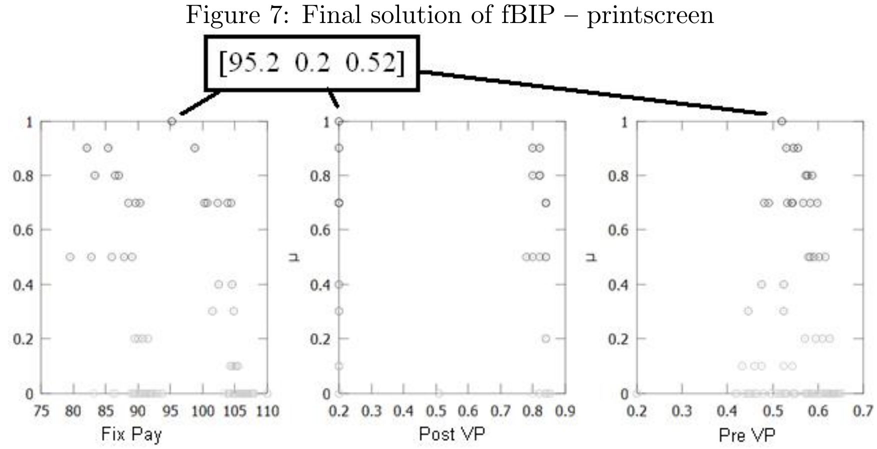
$$C_{[0,9]} = \begin{bmatrix} 0,1 & 1,86 & 0,92 \\ 0,1 & 0 & 0 \\ 0 & 0 & 0,92 \end{bmatrix},$$

$$\left({}_{-}A_{[1]}\right)^T = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & -1 & 0 & 0 & 1 & 40 \end{bmatrix},$$

$$\left({}^{-}b_{[1]}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 100 & 0,8 & 0,8 & 120 \end{bmatrix}.$$

Automatic application of the path I^+, I^-, I^0 to the problem $\mathrm{MOLP}_{\alpha=0,9}$ gives the solution $\mathrm{x}_T(0,9) = \left[\begin{array}{ccc} 82 & 0,8 & 0,55 \end{array}\right]^T$ accepted by the decision maker at the level of $\alpha = 0,9$, accordingly we can find the vector $\left[\begin{array}{ccc} 88,54 & 0,84 & 0,57 \end{array}\right]^T$ with the membership 0,7.
The final solution – set of vectors resulting from the series of MOLP problems – is presented to the decision maker, see Figure 7. At the membership $\alpha = 1$ there is a unique solution of fixed postpaid payment 95,2 PLN, one minute charge in postpaid – 0,20 PLN, and in prepaid – 0,52 PLN. At lower membership values there is a few points and the decision maker might observe their distribution.
The problem of telecommunication services pricing is formulated as the fuzzy

Figure 7: Final solution of fBIP – printscreen



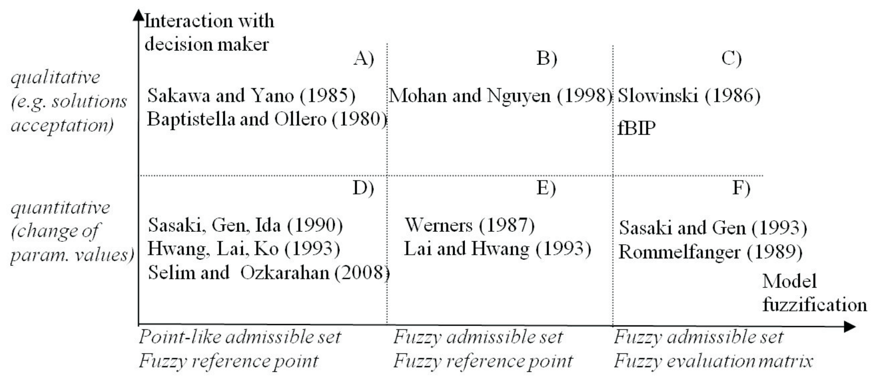
Notes: The final solution is mapped into two-dimensional spaces spanned by the decision variable and the membership value. The color depth is correlated with the membership value. In the picture the singleton of membership $\alpha = 1$ is depicted, $\mathrm{x} = \left[\begin{array}{ccc} 95,2 & 0,2 & 0,55 \end{array}\right]^T$.

multiobjective linear programming problem, bacause its primary charactrization includes non-point elements. It is solved with fuzzy bireference procedure, where the decision maker qualitatively evaluates the subsequent trial solutions choosing the criteria to be bettered, worsened or left unchanged - for details see the bireference procedure by Michalowski and Szapiro (1992). These questions seem to be natural and easy to answer even for the users without any mathematical background. On the other hand the final solution is efficient and consistent with the decision maker preferences.

5 Discussion

To analyse the properties of presented method let us review the fuzzy interactive methods. Baptistella and Ollero (1980) introduced fuzzy procedure for decision support with point-like admissible set. The optimization problem is formulated with point-like matrices, but the procedure applies linguistic variables for changing the parameter values. Thus the decision maker preferences have fuzzy representation.
Sakawa and Yano (1985) just like Baptistella and Ollero (1980) consider the point-like admissible set, but they assume existence of the reference point y^U in the evaluation space. Moreover, they assume for every trial evaluation the decision maker is able to present the section $[^-\eta_j, ^+\eta_j]_{j=1\ldots m}$ describing the minimal and maximal membership of the recommendation to the reference ideal point. Manipulation with the section $[^-\eta_j, ^+\eta_j]_{j=1\ldots m}$ influences the criteria weights in the Chebysheff function scalarizing the criteria values.
The point-like set of admissible solutions is also considered by Sasaki, Gen, Ida (1990), Hwang, Lai, Ko (1993) and Selim and Ozkarahan (2008). Selim and Ozkarahan (2008) assume that the reference point is fuzzy and the interaction is performed by the modification of the membership function in the reference point. Sasaki, Gen, Ida (1990) and Hwang, Lai, Ko (1993) assume, there is a point-like reference evaluation and a fuzzy divergence between the reference point and the evaluation of the current trial solution. If the decision maker modifies the reference point, then new trial solution is found. The other important feature expanding the application field of the optimization model employed by Sasaki, Gen, Ida (1990) and Hwang, Lai, Ko (1993) is the linear structure to describe the decision problem.
The presented procedures employ the point-like linear structure and the fuzzy reference. In the following we describe procedures using also fuzzy parameters of the optimization model. Werners (1987) and Lai and Hwang (1992) consider the fuzzy constraints and assume the decision maker to manipulate it and the fuzzy reference point. Mohan and Nguyen (1998) extend the fuzziness on the criterion matrix. In this approach the interaction is completed by the manipulation of reference membership grade. Choosing of the grade reduces the fMOLP problem to the MOLP problem and enables application of the point-like procedure by Korhonen and Laakso (1986).
The third group of the interactive optimization procedures employs linear structure but all the parameters: the technological matrix, the constraint, the criteria matrix, and the reference point, remain fuzzy. In particular, Slowinski (1986), Rommelfanger (1989) and Sasaki and Gen (1993) assume the parameters to be LR fuzzy numbers. During the optimization process Rommelfanger (1989) proposes to manipulate all the parameters but gives some rules for these operations. Sasaki and Gen (1993) introduce a procedure where only the reference point is manipulated (extension of the Zionts-Wallenius method II, 1983). Slowinski (1986) proposes interaction with the decision maker only at the level of acceptance or rejection of some trial solutions (extension of the procedure by Choo and Atkins, 1980). The result of the interactive optimization methods review is presented in Figure 8. The methods are

Figure 8: Fuzzy interactive decision support



Notes: Map of the interactive fuzzy optimization methods considering number of fuzzified parameters and the interaction type.

divided according to two criteria: type of interaction and type of fuzzification. The quantitative interaction means using the numeric values to restructurize the decision problem by DM, see D,E,F in Figure 8. The qualitative interaction is performed through acceptance/rejection of the recommendations or choosing the most preferred between presented trial solutions, see A,B,C in Figure 8. Three types of fuzzification are considered here. The first class of models exploits point-like feasible set and only the reference point is fuzzy, see A,D in Figure 8. The second class next to the fuzzy reference admits also the fuzzy feasible set, see B,E in Figure 8. The third class expands the fuzzy model from B,E by application of fuzzy evaluation matrix, see C,F in Figure 8.

Basing on the two criteria – type of interaction and type of fuzzification – the procedure by Slowinski (1986) and fBIP have the most interesting properties, because the fuzzy admissible set and fuzzy evaluation matrix are the most flexible structure from the application point of view and the qualitative interaction style is available even for the decision makers with introductory mathematical competences. The methods differ significantly in the interaction style. In every iteration of the optimization procedure Slowinski (1986) assumes the DM will choose the most preferred solution between the presented trial points, where the number of points depends on the number of criteria. In fBIP procedure the DM for single trial solution chooses the criteria to be bettered, worsened or left unchanged. The second important difference between the methods is in the form of final solution. The procedure by Slowinski (1986) provides only one singleton, while fBIP procedure leads to fuzzy set of plenty singletons at various levels of membership function.

6 Concluding remarks

In this paper we considered a problem of revealing the rationality in situations requiring the non-point-like and subjective description. The motivations for introduction of non point-like description are: physiological, psychological and economical, and they are generally associated with inertness of human senses. Moreover, there are decision situations, where various decision makers choose different recommendations. Therefore, next to some objective information we also considered some subjective perspective of the problem, especially in goal formulation.

For the decision problems requiring the non-point-like and subjective description we used the fuzzy multiobjective linear programming. The fuzzy set theory considers the non-point-like representation, while the multiobjective framework regards the subjective evaluation of the decisions.

We proposed to apply the reduced fuzzy matrices where the support is reduced to single line in $\mathbb{R}^n$ space but it retains the economical interpretation and has interesting theoretical properties. In particular, for the fuzzy vectors $[a_i]$ and the reduced fuzzy matrix A of the same membership function and nadir-zenit points the result of multiplication by fuzzy vector x is equal. Moreover, the solution of fMOLP problem defined by reduced fuzzy matrices A,b,C is equal to sum of solution in the series of fMOLP problems with fuzzy A,b and point-like matrix C, and even it is equal to sum of MOLP problems solutions.

The decomposability of the fMOLP problem enables application of the methods and theorems formulated for point-like problems. In particular, we proposed employing the bireference procedure to solve the fuzzy multiobjective linear problems. The method requires the standard linear structure of the optimization model and it operates with qualitative opinions in the interactive phase.

The functioning of the fuzzy bireference procedure was presented on the problem of pricing the telecommunication services. The practical example shows it is quite easy to formulate the fMOLP problem with reduced parameters. Moreover, we observed that the linear splines are flexible enough to represent a wide range of membership functions.

In this paper we did not consider the numerical complexity of the optimization algorithm, but we claim it is polynomial, because the complexity of point-like bireference procedure is about $O(k^3)$ and it is calculated only in the spline points.

We also did not present the method for the membership functions elicitation. This phase is crucial for practical problem solving, but we purposely left it for detailed studies of human cognition. We expect here the growing role of dynamic animations and voice recognition systems.

In future works the authors are going to analyze the sensitivity of the procedure result to the membership function shape and eventually identification of the class broader then the reduced fuzzy numbers, where the decomposition principle holds.

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Appendix

The Appendix includes the proofs of theorems introduced in the paper.

A Proof of Theorem 1

Let us consider the singleton $S(\mathrm{x},\alpha) \in \mathcal{F}(\mathbb{R}^n_+)$ solving the set of inequalities. Then

$$\forall_{j=1\ldots k}\, \exists_{\mathrm{a}_j \in a_{j[\alpha]},\, \mathrm{b}_j \in b_{j[\alpha]}}\ \mathrm{a}_j^T \cdot \mathrm{x} \leq \mathrm{b}_j \Leftrightarrow$$
$$\forall_{j=1\ldots k}\ {}_{-}\mathrm{a}_j^T \cdot \mathrm{x} \leq^{-} \mathrm{b}_{j[\alpha]} \Leftrightarrow_{-} A_{[\alpha]} \cdot \mathrm{x} \leq^{-} b_{[\alpha]} \Leftrightarrow A \cdot S(\mathrm{x},\alpha) \leq b.$$

□

B Proof of Theorem 2

(T1) If the singleton $S(\mathrm{x}^*,\gamma)$ is efficient in fMOLP$_{AbC}$, then it is member of the admissible set X_{Ab} at the level of α, $\alpha \geq \gamma$, and there exist a criteria matrix C of the membership ε, $\varepsilon \geq \gamma$, that the solution x^* has nondominated evaluation and $\gamma = \min(\alpha,\varepsilon)$. Then the singleton $S(\mathrm{x}^*,\alpha)$ is efficient also in fMOLP$_{Ab\mathrm{C}}$ problem. The inference holds in both directions, that ends the proof:

$$(\mathrm{x}^*,\gamma) \in X^E_{AbC} \Leftrightarrow$$
$$\gamma = \min(\alpha,\varepsilon) \wedge \mathrm{x}^* \in X_{Ab[\alpha]} \wedge \exists_{\mathrm{C}\in C[\varepsilon]} \neg\exists_{\mathrm{x}\in X_{Ab[\alpha]},\mathrm{x}\neq\mathrm{x}^*}\ \mathrm{C}\cdot\mathrm{x}^* \leq \mathrm{C}\cdot\mathrm{x} \Leftrightarrow$$
$$\exists_{\mathrm{C}\in C_{[\varepsilon]}}\ (\mathrm{x}^*,\alpha) \in X^E_{Ab\mathrm{C}}.$$

(T2) If the singleton $S(\mathrm{x}^*,\gamma)$ is efficient in fMOLP$_{AbC}$, then there exist the matrix $\mathrm{A} \in_{-} A[\varepsilon]$ and the vector $\mathrm{b} \in b[\alpha]$, that $\gamma = \min(\alpha,\varepsilon)$ and $\mathrm{A}\cdot\mathrm{x} \leq \mathrm{b}$. Therefore the singleton $S(\mathrm{x}^*,\gamma)$ is efficient in the fMOLP problem $Z^F_{\mathrm{A}bC}$, $\mathrm{A} \in \mathrm{supp}({}_{-}A[\varepsilon])$. The inference holds in both directions, and the admissible sets of fMOLP$_{AbC}$ and $\{\text{fMOLP}_{\mathrm{A}bC}\}_{\mathrm{A}\in\mathrm{supp}(_{-}A[\varepsilon])}$ include respective elements, what at the same criteria matrix $C \in \mathcal{F}(\mathbb{R}^{m\times n})$ ends the proof:

$$(\mathrm{x}^*,\gamma) \in X_{Ab} \Leftrightarrow \gamma = \min(\alpha,\varepsilon) \wedge \mathrm{x}^* \in X_{\mathrm{A}b[\alpha]} \wedge \mathrm{A} \in_{-} A_{[\varepsilon]}.$$

(T3) If the singleton $S(\mathrm{x}^*,\gamma)$ is efficient in fMOLP$_{AbC}$, then there exist the matrix $\mathrm{A} \in A[\alpha]$ and the vector $\mathrm{b} \in^{-} b[\varepsilon]$, that $\gamma = \min(\alpha,\varepsilon)$ and $\mathrm{A}\cdot\mathrm{x} \leq \mathrm{b}$. Therefore the singleton $S(\mathrm{x}^*,\gamma)$ is efficient in the fMOLP problem $Z^F_{A\mathrm{b}C}$, $\mathrm{b} \in \mathrm{supp}({}^{-}b[\varepsilon])$. The inference holds in both directions, and the admissible sets of fMOLP$_{AbC}$ and $\{\text{fMOLP}_{AbC}\}_{\mathrm{b}\in\mathrm{supp}(^{-}b[\varepsilon])}$ include respective elements, what at the same criteria matrix $C \in \mathcal{F}(\mathbb{R}^{m\times n})$ ends the proof:

$$(\mathrm{x}^*,\gamma) \in X_{Ab} \Leftrightarrow \gamma = \min(\alpha,\varepsilon) \wedge \mathrm{x}^* \in X_{A\mathrm{b}[\varepsilon]} \wedge \mathrm{b} \in^{-} b_{[\alpha]}.$$

□